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New methods for the two-dimensional Schrödinger equation: SUSY-separation of variables and shape invariance

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Abstract

Two new methods for the investigation of two-dimensional quantum systems, whose Hamiltonians are not amenable to separation of variables, are proposed. The first one—SUSY-separation of variables—is based on the intertwining relations of higher-order SUSY quantum mechanics (HSUSY QM) with supercharges allowing separation of variables. The second one is a generalization of shape invariance. While in one dimension shape invariance allows us to solve algebraically a class of (exactly solvable) quantum problems, its generalization to higher dimensions has not been explored yet. Here we provide a formal framework in HSUSY QM for two-dimensional quantum mechanical systems for which shape invariance holds. Given the knowledge of one eigenvalue and eigenfunction, shape invariance allows us to construct a chain of new eigenfunctions and eigenvalues. These methods are applied to a two-dimensional quantum system, and partial explicit solvability is achieved in the sense that only part of the spectrum is found analytically and a limited set of eigenfunctions is constructed explicitly.

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1. Introduction

In one-dimensional quantum mechanics the importance of exactly solvable and quasi-exactly solvable (QES) models has been stressed repeatedly. The approach of supersymmetric quantum mechanics (SUSY QM) and, in particular, shape invariance [1] has been fully exploited for construction and investigation of such models by generating a partnership between pairs of dynamical systems which allows us to establish the solvability of one in

terms of the other by means of intertwining relations with supercharges of first order in derivatives.

With this knowledge one can construct a variety of multidimensional solvable quantum systems by suitable combination of solvable one-dimensional dynamical systems (separation of variables and its generalizations [2]). From now on we will focus our main attention on two-dimensional quantum systems without assuming that such separation of variables is possible. In two-dimensional quantum mechanics, solvable (or partially solvable) dynamical systems, for which the entire spectrum (or part of it) and the associated wavefunctions are known, play a role similar to solvable (or QES) models in one dimension.

Within the search for a larger class of problems which can be solved by supersymmetrical methods, extensions of SUSY QM have been elaborated with different realizations of the intertwining operators (supercharges). In particular, one-dimensional supercharges were constructed in terms of higher derivative operators [3, 4] (HSUSY QM), the associated superalgebra (HSUSY) containing a higher order polynomial of the Hamiltonian. This generalization has been revisited recently in [5–7] and referred to as N-fold SUSY. Its most simplified version [5] (A-type N-fold SUSY) corresponds to a solvable ansatz for the so-called [3] reducible supercharges and for more general factorized supercharges. However, a class of higher order supertransformations was also found [3], which cannot be represented as a succession of two standard first-order supertransformations (the so called irreducible supercharges). Accordingly, the second-order intertwining operators were exhaustively classified as reducible and irreducible. We warn that our definition of reducibility [3] does not coincide with the factorization of supercharge.

An important step for the investigation of one-dimensional HSUSY models is the construction of zero modes of (higher order) supercharges which can be instrumental for QES [5, 7] and for the study [4] of spectral properties of higher order shape invariant systems. Indeed, it is well known that in one-dimensional standard SUSY QM, a very elegant method of solution of the spectral problem exists for potentials which preserve their shape in the SUSY partnership [1, 8]. It was shown that for a wide variety of such potentials the entire spectrum and the eigenfunctions can be obtained algebraically providing a fresh reformulation of the old factorization method [9] for the Schrödinger equation. Extensions of shape invariance to third order not fully reducible HSUSY QM models were proposed in [4].

In the N-dimensional SUSY QM [10] starting from a scalar Schrödinger Hamiltonian a chain of matrix Hamiltonians of different dimensionality was constructed ending with another scalar Hamiltonian. Each pair of neighbouring Hamiltonians is intertwined by first-order supercharges, each Hamiltonian has a partial isospectrality with both neighbours. In the two-dimensional case this chain simplifies to two scalar Hamiltonians and one 2×2 matrix Hamiltonian. The spectra of the two scalar Hamiltonians build up the spectrum of the matrix Hamiltonian, but in general the spectra of the scalar Hamiltonians are not related.

However, cases were found [11, 12] where the two scalar two-dimensional Hamiltonians are intertwined by second-order supercharges and are therefore isospectral, up to zero modes of the supercharges. This suggests that HSUSY QM may even be more important for the two-dimensional case than for the one-dimensional case to study the spectra and the eigenfunctions of two-dimensional models not amenable to separation of variables. Due to the complexity of the system of nonlinear partial differential equations, arising from the second-order intertwining relations, one has to look only for particular solutions. Indeed, two classes of such particular systems were found in [11, 12].

In section 2 we outline the second-order SUSY QM framework in one and two dimensions, which allows us to formulate in section 3 the method of SUSY-separation of variables for two-dimensional Hamiltonians, which are not amenable to separation of variables. This

method can be used when the *supercharges* allow for separation of variables. In section 4 a two-dimensional singular Morse-type potential (subsection 4.1) is shown to satisfy SUSY-separation of variables. For this model the zero modes of the supercharge (subsection 4.2) and eigenfunctions of the Hamiltonian (subsection 4.5) in the linear space of zero modes of the supercharge are constructed. In addition, a new eigenfunction outside this space is built explicitly (subsection 4.6). The role of shape invariance (subsection 5.1) has not been explored yet in the two-dimensional SUSY QM (subsection 5.2). One aim of this paper (subsection 5.3) is to find the Hamiltonians (without separation of variables) which realize the two-dimensional generalization of shape invariance. The model of subsection 4.1 is found to satisfy shape invariance. It is shown that, due to the non-trivial space of zero modes of supercharges, shape invariance does not allow a fully algebraic solution for the entire spectrum, but only (partial explicit solvability) for a part of the spectrum and for the corresponding wavefunctions. Section 6 contains a brief discussion of the two-dimensional Morse-type model in another region of parameter values and the integrability of this model, by elucidating the action of quantum integrals of motion, commuting with the Hamiltonian, on the constructed eigenfunctions.

2. Second-order SUSY in one and two dimensions

Let us provide notations and the main formulae for standard SUSY QM [1]:

$$\tilde{H} = Q^+Q^- = -\partial^2 + \tilde{V}(x) \qquad \tilde{H}\tilde{\Psi}_n(x) = E_n\tilde{\Psi}_n(x) \tag{1}$$

$$H = Q^-Q^+ = -\partial^2 + V(x) \qquad H\Psi_n(x) = E_n\Psi_n(x) \tag{2}$$

$$\tilde{H}Q^+ = Q^+H \qquad Q^-\tilde{H} = HQ^- \tag{3}$$

$$Q^+ = -\partial + W(x) \qquad Q^- = (Q^+)^\dagger = \partial + W(x) \tag{4}$$

$$\Psi_n(x) = Q^-\tilde{\Psi}_n(x) \qquad \tilde{\Psi}_n(x) = Q^+\Psi_n(x) \tag{5}$$

where $\partial \equiv \mathrm{d}/\mathrm{d}x$.

The spectral equivalence (up to zero modes of $Q^\pm$) of $\tilde{H}, H$ can be expressed via the superalgebra

$$\hat{H} = \begin{pmatrix} \tilde{H} & 0 \\ 0 & H \end{pmatrix} \qquad \hat{Q}^+ = (\hat{Q}^-)^\dagger = \begin{pmatrix} 0 & 0 \\ Q^- & 0 \end{pmatrix}$$
$$\{\hat{Q}^+, \hat{Q}^-\} = \hat{H} \qquad (\hat{Q}^+)^2 = (\hat{Q}^-)^2 = 0 \qquad [\hat{H}, \hat{Q}^\pm] = 0. \tag{6}$$

The second-order generalization of the superalgebra incorporates [3] the most general intertwining operators of second order in derivatives:

$$Q^+ = \partial^2 - 2f(x)\partial + b(x) \qquad Q^- = (Q^+)^\dagger. \tag{7}$$

Intertwining relations (3) with supercharges (7) lead [3] to the expressions of the potentials $V, \tilde{V}$ and the supercharges $Q^\pm$ in terms of the only real function $f(x)$:

$$\tilde{V}(x) = -2f'(x) + f(x)^2 + \frac{f''(x)}{2f(x)} - \left(\frac{f'(x)}{2f(x)}\right)^2 - \frac{d}{4f(x)^2} - a$$

$$V(x) = 2f'(x) + f(x)^2 + \frac{f''(x)}{2f(x)} - \left(\frac{f'(x)}{2f(x)}\right)^2 - \frac{d}{4f(x)^2} - a$$

$$b(x) = -f'(x) + f(x)^2 - \frac{f''(x)}{2f(x)} + \left(\frac{f'(x)}{2f(x)}\right)^2 + \frac{d}{4f(x)^2}$$

where a and d are arbitrary real constants. The case $d \leqslant 0$ was called [3] reducible, since it can be interpreted in terms of two successive first-order supertransformations with real

superpotentials and a Hermitian intermediate Hamiltonian (up to a constant real energy shift). The alternative case $d > 0$ was called irreducible.

A two-dimensional generalization of standard SUSY QM was proposed in [10], where the superHamiltonian includes a chain of one matrix and two scalar Hamiltonians. Each scalar Hamiltonian is separately intertwined with the matrix Hamiltonian, but the spectra of the two scalar Hamiltonians are not related.

If one wants to relate directly [11, 12] the spectra of two scalar two-dimensional Schrödinger operators via intertwining relations analogous to (3):

$$\begin{aligned} &\tilde{H}(\vec{x})Q^+ = Q^+H(\vec{x}) \qquad Q^-\tilde{H}(\vec{x}) = H(\vec{x})Q^- \\ &H = -\triangle + V(\vec{x}) \qquad \tilde{H} = -\triangle + \tilde{V}(\vec{x}) \qquad \triangle \equiv \partial_1^2 + \partial_2^2 \qquad \partial_i \equiv \partial/\partial x_i \end{aligned} \tag{8}$$

it is expedient to make use of a two-dimensional generalization of (7):

$$Q^+ = (Q^-)^\dagger = g_{ik}(\vec{x})\partial_i\partial_k + C_i(\vec{x})\partial_i + B(\vec{x}) \tag{9}$$

where all coefficient functions are real.

This means that, up to zero modes of $Q^\pm$, the spectra of $H, \tilde{H}$ coincide and their eigenfunctions are mutually connected:

$$\Psi \sim Q^-\tilde{\Psi} \qquad \tilde{\Psi} \sim Q^+\Psi.$$

A classification of the dynamical systems requires first to determine the 'metrics' in supercharges (7). Equation (8) leads to a set of four equations for the metrics $g_{ik}(\vec{x})$

$$\partial_i g_{kl} + \partial_k g_{il} + \partial_l g_{ik} = 0 \qquad \{ikl\} = \{111; 112; 122; 222\} \tag{10}$$

which can be solved independently from the other ones:

$$\begin{aligned} &g_{11} = ax_2^2 + a_1x_2 + b_1 \qquad g_{22} = ax_1^2 + a_2x_1 + b_2 \\ &g_{12} = -\tfrac{1}{2}(2ax_1x_2 + a_1x_1 + a_2x_2) + b_3 \end{aligned}$$

with a, a_i, b_i constants.

For the particular case of the unit metrics $g_{ik} = \delta_{ik}$ the corresponding quantum systems allow [11] for the so-called [2] R-separation of variables in parabolic, elliptic or polar coordinate systems.

After having solved (10), the intertwining relations (8) are equivalent to the following system of differential equations for potentials $V(\vec{x}), \tilde{V}(\vec{x})$ and coefficient functions $C_k(\vec{x}), B(\vec{x})$:

$$\begin{aligned} &\partial_i C_k + \partial_k C_i + \triangle g_{ik} - (\tilde{V} - V)g_{ik} = 0 \\ &\triangle C_i + 2\partial_i B + 2g_{ik}\partial_k V - (\tilde{V} - V)C_i = 0 \\ &\triangle B + g_{ik}\partial_k\partial_i V + C_i\partial_i V - (\tilde{V} - V)B = 0. \end{aligned} \tag{11}$$

Because of the complexity of the above equations, only particular solutions have been found in [11, 12] by making suitable ansatzes. In particular, for the supercharges with *Lorentz metrics* ($g_{ik} = \mathrm{diag}(1, -1)$):

$$Q^+ = \left(\partial_1^2 - \partial_2^2\right) + C_k\partial_k + B = 4\partial_+\partial_- + C_+\partial_- + C_-\partial_+ + B \tag{12}$$

a solution of (11) can be reduced [11, 12] to a solution of the system:

$$\partial_-(C_-F) = -\partial_+(C_+F) \tag{13}$$

$$\partial_+^2 F = \partial_-^2 F \tag{14}$$

where $x_\pm \equiv x_1 \pm x_2$, $\partial_\pm = \partial/\partial x_\pm$ and $C_\pm$ depend only on $x_\pm$, respectively:

$$C_+ \equiv C_1 - C_2 \equiv C_+(x_+) \qquad C_- \equiv C_1 + C_2 \equiv C_-(x_-).$$

The function F, solution of (14), is represented as

$$F = F_1(x_+ + x_-) + F_2(x_+ - x_-).$$

The potentials $\tilde{V}(\vec{x})$, $V(\vec{x})$ and the function $B(\vec{x})$ are expressed in terms of $F_1(2x_1)$, $F_2(2x_2)$ and $C_\pm(x_\pm)$, solutions of system (13), (14):

$$\begin{aligned} \tilde{V} &= \tfrac{1}{2}(C'_+ + C'_-) + \tfrac{1}{8}(C_+^2 + C_-^2) + \tfrac{1}{4}(F_2(x_+ - x_-) - F_1(x_+ + x_-)) \\ V &= -\tfrac{1}{2}(C'_+ + C'_-) + \tfrac{1}{8}(C_+^2 + C_-^2) + \tfrac{1}{4}(F_2(x_+ - x_-) - F_1(x_+ + x_-)) \end{aligned} \tag{15}$$

$$B = \tfrac{1}{4}(C_+C_- + F_1(x_+ + x_-) + F_2(x_+ - x_-)). \tag{16}$$

3. SUSY-separation of variables: a construction of new two-dimensional partially solvable models

We want to study two-dimensional Hamiltonians without separation of variables within the SUSY approach. Our goal is to find a class of Hamiltonians for which part of the spectrum and the eigenfunctions can be found (partially solvable systems). Our approach is based on the solution of the intertwining relations (8) for partner potentials and supercharges followed by the investigation of zero modes of the supercharges (7). Because, in principle, there is no connection between separation of variables in the supercharges and the Hamiltonians, one can look for the opportunity of partially solving the spectrum of a Hamiltonian (which does not allow the separation of variables) via normalizable zero modes of supercharges which do allow such separation.

More specifically we will start from the investigation of zero modes of supercharges which do allow separation of variables and solve the intertwining relations (8) obtaining Hamiltonians which are not amenable to separation of variables. The algebraic method (for partial solution) will be presented in this section in its general form. In section 4 the method will be applied to a specific type of two-dimensional models.

Let us suppose that there are $(N + 1)$ normalizable zero modes $\Omega_n(\vec{x})$, $n = 0, 1, \ldots, N$ of the supercharge Q^+:

$$Q^+\vec{\Omega}(\vec{x}) = 0 \tag{17}$$

where $\vec{\Omega}(\vec{x})$ is a column vector with components $\Omega_n(\vec{x})$. By intertwining (8) onto $\vec{\Omega}(\vec{x})$ it is easy to realize that the space of zero modes is closed under the action of H:

$$H\vec{\Omega}(\vec{x}) = \hat{C}\vec{\Omega}(\vec{x}) \tag{18}$$

where $\hat{C} \equiv \|c_{ik}\|$ is a c-number $\vec{x}$-independent real matrix. If the matrix $\hat{C}$ can be diagonalized[4] by a real similarity transformation:

$$\hat{B}\hat{C}(\hat{B})^{-1} = \hat{\Lambda} = \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_N) \tag{19}$$

the problem reduces to a standard algebraic task within the zero modes space:

$$H(\hat{B}\vec{\Omega}(\vec{x})) = \hat{\Lambda}(\hat{B}\vec{\Omega}(\vec{x})). \tag{20}$$

To attack the problem it is expedient to eliminate the first-order derivative terms in the supercharge (12) by a suitable similarity transformation:

$$q^+ = \exp(-\chi(\vec{x}))Q^+\exp(+\chi(\vec{x})) = \partial_1^2 - \partial_2^2 + \tfrac{1}{4}(F_1(2x_1) + F_2(2x_2)) \tag{21}$$

$$\chi(\vec{x}) = -\frac{1}{4}\left(\int C_+(x_+)\,\mathrm{d}x_+ + \int C_-(x_-)\,\mathrm{d}x_-\right). \tag{22}$$

[4] If $\hat{C}$ cannot be diagonalized, the algebraic method can be applied if one can solve the less restrictive matrix equation $\hat{B}\hat{C} = \hat{\Lambda}\hat{B}$.

We note that q^+ exhibits separation of variables: this is what we mean by SUSY-*separation of variables*, even if the components of the superHamiltonian do not admit such separation. Zero modes of q^+ can be found as a linear superposition of products of one-dimensional wavefunctions $\eta_n(x_1)$ and $\rho_n(x_2)$, satisfying the Schrödinger equations:

$$\begin{aligned}\left(-\partial_1^2 - \tfrac{1}{4}F_1(2x_1)\right)\eta_n(x_1) &= \epsilon_n\eta_n(x_1)\\ \left(-\partial_2^2 + \tfrac{1}{4}F_2(2x_2)\right)\rho_n(x_2) &= \epsilon_n\rho_n(x_2)\end{aligned} \tag{23}$$

with ϵ_n the separation constants.

In analogy to (21), one can define operators

$$h \equiv \exp(-\chi(\vec{x}))H\exp(+\chi(\vec{x})) = -\partial_1^2 - \partial_2^2 + C_1(\vec{x})\partial_1 - C_2(\vec{x})\partial_2 - \tfrac{1}{4}F_1(2x_1) + \tfrac{1}{4}F_2(2x_2) \tag{24}$$

$\tilde{h}$ and eigenfunctions of q^+ as

$$\omega_n(\vec{x}) = \exp(-\chi(\vec{x}))\cdot\Omega_n(\vec{x}) \tag{25}$$

keeping, however, in mind that the normalizability and orthogonality are not preserved automatically due to non-unitarity of the similarity transformation. The operators h and $\tilde{h}$ should not be interpreted literally as Hamiltonians since they are non-Hermitian (but have real spectrum)[5] and are not factorized in $q^{\pm}$, as in (1) and (2).

Then using (23) one can write

$$h\omega_n(\vec{x}) = [2\epsilon_n + C_1(\vec{x})\partial_1 - C_2(\vec{x})\partial_2]\omega_n(\vec{x}). \tag{26}$$

While from the above equation it is not manifest, we know, however, from (18) that the space spanned by functions $\omega_n(\vec{x})$ is closed under the action of h. In the concrete model of section 4 this will be demonstrated in subsection 4.4.

It is clear that, in contrast to (21) where variables are separated, there is no separation of variables for h, which makes the two-dimensional dynamics not-trivially reducible to one-dimensional dynamics. For this reason we refer to this method for partial solvability as *SUSY-separation of variables*.

4. A new two-dimensional partially solvable model

4.1. Formulation of the model

In this section we apply the method of section 3 to a particular example, which can be interpreted as a suitable two-dimensional generalization of the Morse potential. Among the solutions of the system of equations (13), (14) found in [12] we focus our attention on the particular case already presented in [12]:

$$\begin{aligned}F_1(x) &= k_1\left(\alpha_+\alpha_-\exp(\lambda x) + \beta_+\beta_-\exp(-\lambda x)\right) + k_2(\alpha_+^2\alpha_-^2\exp(2\lambda x) + \beta_+^2\beta_-^2\exp(-2\lambda x))\\ -F_2(x) &= k_1\left(\alpha_+\beta_-\exp(\lambda x) + \beta_+\alpha_-\exp(-\lambda x)\right) + k_2(\alpha_+^2\beta_-^2\exp(2\lambda x) + \beta_+^2\alpha_-^2\exp(-2\lambda x))\\ C_\pm &= \pm\frac{\alpha_\pm\exp(\lambda x_\pm) + \beta_\pm\exp(-\lambda x_\pm)}{\lambda\left(\alpha_\pm\exp(\lambda x_\pm) - \beta_\pm\exp(-\lambda x_\pm)\right)}\end{aligned} \tag{27}$$

[5] Incidentally, we remark that by an additional unitary transformation one can generate analytically a class of two-dimensional non-trivial Hamiltonians with complex potentials but with real spectrum. Examples of two-dimensional Hamiltonians have been discussed recently in [14].

$$V = \frac{2\alpha_+\beta_+(1+8\lambda^2) + \alpha_+^2\exp(2\lambda x_+) + \beta_+^2\exp(-2\lambda x_+)}{8\lambda^2(\alpha_+\exp(\lambda x_+) - \beta_+\exp(-\lambda x_+))^2}$$
$$+ \frac{2\alpha_-\beta_-(1-8\lambda^2) + \alpha_-^2\exp(2\lambda x_-) + \beta_-^2\exp(-2\lambda x_-)}{8\lambda^2(\alpha_-\exp(\lambda x_-) - \beta_-\exp(-\lambda x_-))^2}$$
$$- \tfrac{1}{4}\big[k_1(\alpha_+\beta_-\exp(2\lambda x_2) + \alpha_-\beta_+\exp(-2\lambda x_2)) + k_2\big(\alpha_+^2\beta_-^2\exp(4\lambda x_2)$$
$$+ \alpha_-^2\beta_+^2\exp(-4\lambda x_2)\big) + k_1(\alpha_+\alpha_-\exp(2\lambda x_1) + \beta_+\beta_-\exp(-2\lambda x_1))$$
$$+ k_2\left(\alpha_+^2\alpha_-^2\exp(4\lambda x_1) + \beta_+^2\beta_-^2\exp(-4\lambda x_1)\right)\big]. \tag{28}$$

We will consider a specific case of the expressions for $C_\pm$ and V by choosing the parameters in (27) and (28) so that $\beta_+ = 0, \alpha_- = \beta_-, \lambda \equiv -\alpha/2, \alpha > 0$. It is evident that equation (13) admits a more general solution than (27) because one can introduce an additional multiplicative parameter in $C_\pm$ in (27). The nonlinear dependence of (15) on $C_\pm(x_\pm)$ leads to a non-trivial generalization of the corresponding potential.

We then obtain

$$C_+ = 4a\alpha \qquad C_- = 4a\alpha\coth\frac{\alpha x_-}{2} \tag{29}$$

$$f_1(x_1) \equiv \tfrac{1}{4}F_1(2x_1) = -A(\exp(-2\alpha x_1) - 2\exp(-\alpha x_1)) \tag{30}$$

$$f_2(x_2) \equiv \tfrac{1}{4}F_2(2x_2) = +A(\exp(-2\alpha x_2) - 2\exp(-\alpha x_2)) \tag{31}$$

$$\tilde{V}(\vec{x}) = \alpha^2 a(2a-1)\sinh^{-2}\left(\frac{\alpha x_-}{2}\right) + 4a^2\alpha^2$$
$$+ A[\exp(-2\alpha x_1) - 2\exp(-\alpha x_1) + \exp(-2\alpha x_2) - 2\exp(-\alpha x_2)]$$
$$V(\vec{x}) = \alpha^2 a(2a+1)\sinh^{-2}\left(\frac{\alpha x_-}{2}\right) + 4a^2\alpha^2 \tag{32}$$
$$+ A[\exp(-2\alpha x_1) - 2\exp(-\alpha x_1) + \exp(-2\alpha x_2) - 2\exp(-\alpha x_2)]$$

where A is an arbitrary positive constant and a is a parameter originating from the new multiplicative constant mentioned above. We will show in the following sections that the range of variation of this parameter a will characterize the dynamics of the model. It is perhaps interesting to remark that the reflection $a \to -a$ signals the supertransformation $Q^+ \leftrightarrow Q^-$ and $H \leftrightarrow \tilde{H}$. Both potentials $V(\vec{x})$ and $\tilde{V}(\vec{x})$ are also invariant under the interchange $x_1 \leftrightarrow x_2$ ('x_--parity' conservation). As for standard P-symmetry, this invariance leads to the classification of eigenfunctions according to their 'x_--parity' values.

One easily recognizes in (32) a sum of two Morse potentials plus a hyperbolic singular term which prevents the application of the method of separation of variables for the system (32). These singular terms in $\tilde{V}(\vec{x})$, $V(\vec{x})$ can both be attractive, for the case $|a| > \frac{1}{2}$, or one repulsive and one attractive, for the case $|a| < \frac{1}{2}$. The parameter a will be further constrained by the condition that the strength of the attractive singularity at $x_- \to 0$ should not exceed the well-known bound $-1/(4x_-^2)$.

4.2. Zero modes of Q^+

The reason for the choice of the model (27)–(32) is that the corresponding equations for the zero modes (23) of q^+ can be solved analytically in this case. In particular [13], for the discrete spectrum $\epsilon_n < 0$ the normalizable eigenfunctions are

$$\omega_n(\vec{x}) = \exp\left(-\frac{\xi_1+\xi_2}{2}\right)(\xi_1\xi_2)^{s_n}F(-n, 2s_n+1; \xi_1)F(-n, 2s_n+1; \xi_2) \tag{33}$$

where $F(-n, 2s_n + 1; \xi)$ is the standard degenerate (confluent) hypergeometric function, reducing to a polynomial for integer n, and

$$\xi_i \equiv \frac{2\sqrt{A}}{\alpha} \exp(-\alpha x_i) \tag{34}$$

$$s_n = \frac{\sqrt{A}}{\alpha} - n - \frac{1}{2} > 0 \tag{35}$$

$$\epsilon_n = -A\left[1 - \frac{\alpha}{\sqrt{A}}\left(n + \frac{1}{2}\right)\right]^2. \tag{36}$$

The number $(N + 1)$ of normalizable zero modes (33) is determined by inequality (35). As a final remark, let us point out that the zero modes $\omega_n(\vec{x})$ of q^+ are entirely based on the 'Morse part' of the potentials and consequently do not depend on the parameter a, which reflects the strength of the singular part.

4.3. Normalizability of zero modes of Q^+

We want to discuss the normalizability of the zero modes $\Omega_n(\vec{x})$ which are connected with (33):

$$\Omega_n(\vec{x}) = \exp(\chi(\vec{x}))\omega_n(\vec{x}). \tag{37}$$

While in principle one can investigate the most general conditions for which $\Omega_n(\vec{x})$ is normalizable but $\omega_n(\vec{x})$ is not, we will restrict ourselves, for simplicity, to the conditions for which the normalizability of $\omega_n(\vec{x})$ implies that for $\Omega_n(\vec{x})$. A constructive discussion of these restrictions will be given below.

From (22) and (29) one can find the analytical expression

$$\exp(\chi(\vec{x})) = \exp(-a\alpha x_+)\left|\sinh\left(\frac{\alpha x_-}{2}\right)\right|^{-2a} = \left(\frac{\alpha}{\sqrt{A}}\frac{\xi_1\xi_2}{|\xi_2 - \xi_1|}\right)^{2a} \qquad \alpha > 0. \tag{38}$$

For $\xi_1 \to \xi_2$ the term $|\xi_2 - \xi_1|^{-2a}$ in (38) requires, for the normalizability of (37), that $a < 1/4$.

At infinity (in the $\xi_1 \geqslant 0$, $\xi_2 \geqslant 0$ quadrant) (38) does not change essentially the behaviour of $\omega_n(\vec{x})$, which is $\exp\left(-\frac{\xi_1+\xi_2}{2}\right)$.

At the origin (again in ξ_1, ξ_2) the behaviour of $\omega_n(\vec{x}) \sim (\xi_1\xi_2)^{s_n}$, $s_n > 0$ combines with the behaviour at the origin of (38). In polar coordinates

$$\xi_1 = \xi\cos\phi \qquad \xi_2 = \xi\sin\phi$$

the relevant integrand of $\|\Omega(\vec{x})\|^2$ at the origin reads

$$\mathrm{d}^2x\,\Omega_n^2(\vec{x}) \sim \frac{\mathrm{d}\xi}{\xi}\frac{\mathrm{d}\phi}{\sin(2\phi)}\xi^{4(a+s_n)}\frac{(\sin(2\phi))^{2(2a+s_n)}}{\left|\sin\left(\phi - \frac{\pi}{4}\right)\right|^{4a}}.$$

Combining all these restrictions, normalizability of $\Omega_n(\vec{x})$, therefore, is ensured by

$$s_n + a > 0 \qquad s_n + 2a > 0 \qquad s_n > 0 \qquad a < \tfrac{1}{4}.$$

In addition, one has to impose the constraint that the singularity for the superHamiltonian (i.e. for both H and $\tilde{H}$) should be repulsive or, if attractive, should be bounded, as explained at the end of subsection 4.1, by $4a(2a \pm 1) \geqslant -1/4$. All these constraints can be implemented contextually leading to three-parameter families of models:

$$a \in \left(-\infty, -\frac{1}{4} - \frac{1}{4\sqrt{2}}\right) \qquad s_n = \frac{\sqrt{A}}{\alpha} - n - \tfrac{1}{2} > -2a > 0. \tag{39}$$

Inequalities (39) can be satisfied by appropriate choice of parameters a and A, and/or by suitable restriction of the considered number N of zero modes $\Omega_n(\vec{x})$. It is perhaps interesting to remark that the reflection symmetry $H \leftrightarrow \tilde{H}$, $Q^+ \leftrightarrow Q^-$ for $a \to -a$ is broken by the fact that the operator Q^- has no normalizable zero modes for the parameter range in (39), while Q^+ has. An alternative possible range of parameters which allows us to implement the reflection symmetry explicitly will be discussed in section 6.

4.4. Algebraic solution

In this subsection we will prove explicitly that the operator h, when acting in the linear space of the zero modes of q^+, leaves this space invariant. Inserting (33) into expression (26) one obtains

$$h\omega_n(\vec{x}) = -2\left(2a\alpha^2 s_n - \epsilon_n\right)\omega_n(\vec{x}) + \frac{4a\alpha^2 n}{2s_n+1}\frac{(\xi_1\xi_2)^{s_n+1}}{(\xi_2-\xi_1)}\exp\left(-\frac{1}{2}(\xi_1+\xi_2)\right) \times \Phi(-n+1, 2s_n+2; \xi_1, \xi_2) \tag{40}$$

where the function $\Phi(b, c; \xi_1, \xi_2)$ is defined by

$$\Phi(b, c; \xi_1, \xi_2) \equiv [F(b, c; \xi_1)F(b-1, c-1; \xi_2) - F(b, c; \xi_2)F(b-1, c-1; \xi_1)].$$

Relations between contiguous hypergeometric functions lead to

$$\Phi(b, c; \xi_1, \xi_2) = \frac{b-c}{(c-1)c}(\xi_2-\xi_1)[F(b, c; \xi_1)F(b, c+1; \xi_2) + \frac{b-c}{(c-1)c}\frac{b}{c(c+1)}\xi_1\xi_2\Phi(b+1, c+2; \xi_1, \xi_2). \tag{41}$$

By making repeated use of (41) for $b = 1-n$ one arrives, after laborious manipulations, at

$$\Phi(1-n, c; \xi_1, \xi_2) = (\xi_2-\xi_1)\sum_{k=0}^{n-1} a_{nk}(\xi_1\xi_2)^k\,[F(1-n+k, c+2k+1; \xi_1) \times F(1-n+k, c+2k+1; \xi_2)] \tag{42}$$

where

$$a_{n0} = -\frac{n+c-1}{(c-1)c} \qquad a_{nk} = 0 \qquad \text{for} \quad k > n$$

$$a_{nk} = -\frac{(n-1)!}{k!}\frac{(n+c-1)(n+c)\cdots(n+c+n-k-2)}{[(c-1)(c+2n-2k-2)][c(c+1)\cdots(c+2n-2k-3)]^2} \quad \text{for} \quad k < n.$$

In our case $c = 2s_n + 2$ and (42) becomes

$$\Phi(1-n, 2s_n+2; \xi_1, \xi_2) = (\xi_2-\xi_1)\sum_{k=0}^{n-1} a_{nk}(\xi_1\xi_2)^{n-k-1} \times F(-k, 2s_k+1; \xi_1)F(-k, 2s_k+1; \xi_2).$$

Inserting (42) into (40) and taking into account $s_k - s_n = n-k$ (see (35)), an equation of the type (18) is obtained:

$$h\omega_n(\vec{x}) = \sum_{k=0}^{N} c_{nk}\omega_k(\vec{x}) = -2\left(2a\alpha^2 s_n - \epsilon_n\right)\omega_n(\vec{x}) + \frac{4a\alpha^2 n}{2s_n+1}\sum_{k=0}^{n-1} a_{nk}\omega_k(\vec{x}) \tag{43}$$

showing that the matrix $\hat{C}$ is triangular. We recall (see (37)) that the zero modes $\Omega_n(\vec{x})$ are related to $\omega_n(\vec{x})$ by the similarity transformation.

4.5. Eigenfunctions of H

The triangular matrix $\hat{C}$ with all different and nonzero diagonal elements can be diagonalized by the similarity transformation (19) and its eigenvalues coincide with the diagonal elements c_{kk}. From (20) one can construct a set of eigenfunctions $\psi_n(\vec{x})$ and $\Psi_n(\vec{x})$ of h and H, provided matrix $\hat{B}$ is obtained (see equations (19) and (20)):

$$\psi_{N-n}(\vec{x}) = \sum_{l=0}^{N} b_{nl}\omega_l(\vec{x}) \qquad \Psi_{N-n}(\vec{x}) = \sum_{l=0}^{N} b_{nl}\Omega_l(\vec{x}). \tag{44}$$

The index giving the numeration of the wavefunctions ψ will be elucidated below. The eigenvalues of H which correspond to $\Psi_k(\vec{x})$ are expressed in terms of the parameters of the problem:

$$E_k = c_{kk} = \lambda_{N-k} = -2\left(2a\alpha^2 s_k - \epsilon_k\right). \tag{45}$$

From (45) we conclude that the assumptions made for the diagonal elements of $\hat{C}$ are not very restrictive due to the interplay of the parameters of the problem.

We start from the formal solution for the matrix elements of $\hat{B}$:

$$b_{m,p} = b_{m,N-m}\left[\sum_{l=1}^{N-p-1}\left(\tau^{(m)}\right)^l\right]_{N-m,p} \tag{46}$$

where the $(N+1)$ triangular matrices labelled $\tau^{(m)}$, $m = 0, 1, ..., N$, are defined via the matrix elements of $\hat{C}$:

$$\tau^{(m)}_{n,k} \equiv \frac{c_{n,k}}{c_{N-m,N-m} - c_{k,k}}.$$

We stress that in (46) the expression $\left(\tau^{(m)}\right)^l$ means the lth power of the matrix $\tau^{(m)}$. The repeated index $(N-m)$ is frozen in (46) and not summed over. This expression allows us to write all elements of the mth row $b_{m,p}$ in terms of the matrix $\tau^{(m)}$ and the arbitrary value of the element $b_{m,N-m}$ on the *crossed* diagonal. These arbitrary values can be fixed by the normalization condition for the wavefunctions $\Psi_{N-n}(\vec{x})$ in (44).

From the triangularity of $\hat{C}$ and $\tau^{(m)}$ in (46) it follows that nonzero elements b_{mp} are obtained only for $m + p \leqslant N$. This means the matrix $\hat{B}$ vanishes below the *crossed* diagonal. It is now clear that the index of the wavefunction in (44) is taken in a way to make $\Psi_k(\vec{x})$ a linear combination of the first $(k+1)$ zero modes $\Omega_l(\vec{x}); l = 0, 1, \ldots, k$. In particular, $\Psi_0(\vec{x}) \sim \Omega_0(\vec{x})$.

We sketch a constructive algorithm to justify the result (46) for the matrix $\hat{B}$. It is not difficult to see that indeed the system of equations

$$\sum_{k=0}^{N} b_{i,k}c_{k,l} = \lambda_i b_{i,l}$$

is solved for $i = 0, l = N$ by (46) with $\lambda_0 = c_{N,N}$ and arbitrary $b_{0,N} \neq 0$ due to the triangularity of $\hat{C}$. One can also check that the solution holds for $i = 0$ and $l = (N-1), (N-2), \ldots, 0$ by solving iteratively the corresponding linear equations. For $i = 1$ one convinces oneself that the first equation with our assumptions implies $b_{1,N} = 0$, which signals the crossed triangularity mentioned above. Following the steps as before, $l = (N-1), (N-2), \ldots, 0$, again one can solve iteratively with arbitrary element $b_{1,(N-1)}$. Similar manipulations can be performed for higher values $i = 2, 3, \ldots, N$.

All these wavefunctions $\Psi_k(\vec{x})$ live in the space of zero modes. In the next subsection we will also construct additional eigenfunctions, not linear combinations of zero modes.

4.6. Additional eigenfunctions of H

The eigenfunctions of subsection 4.5 may be used for constructing more general eigenfunctions of h and of H via a product ansatz:

$$\phi(\vec{x}) \equiv \psi_0(\vec{x}) \cdot \Theta(\vec{x}) \qquad \Phi(\vec{x}) \equiv \Psi_0(\vec{x}) \cdot \Theta(\vec{x}). \tag{47}$$

The eigenvalue equation for h leads to

$$L\Theta(\vec{x}) = \gamma\Theta(\vec{x}) \tag{48}$$

where

$$L = -\alpha^2\xi_i^2\partial_{\xi_i}^2 + \alpha^2\xi_i(\xi_i - 2s_0 - 1)\partial_{\xi_i} - 4a\alpha^2\frac{\xi_1\xi_2}{\xi_2 - \xi_1}\left(\partial_{\xi_1} - \partial_{\xi_2}\right).$$

The sum over index i is implicit. More specifically, the eigenvalue equation

$$h\phi(\vec{x}) = c_{00}\phi(\vec{x}) + \psi_0(\vec{x})L\Theta(\vec{x})$$

determines the new eigenvalues

$$H\Phi(\vec{x}) = (E_0 + \gamma)\Phi(\vec{x}). \tag{49}$$

For the spectral problem (48) we can provide only particular solutions by choosing suitable ansatze for the functions $\Theta(\vec{x})$. It is useful to change variables:

$$z_1 = \frac{1}{\xi_1} + \frac{1}{\xi_2} \qquad z_2 = \frac{1}{\xi_1} - \frac{1}{\xi_2}$$

so that

$$L = -\frac{\alpha^2}{2}\left[\left(z_1^2 + z_2^2\right)\left(\partial_{z_1}^2 + \partial_{z_2}^2\right) + 4z_1z_2\partial_{z_1}\partial_{z_2} + 4\partial_{z_1} - 2(2s_0 - 1)\left(z_1\partial_{z_1} + z_2\partial_{z_2}\right)\right.$$
$$\left. - 4a\left(2z_1\partial_{z_1} + z_2\partial_{z_2} + \frac{z_1^2}{z_2}\partial_{z_2}\right)\right].$$

The action of L on monomial products $z_1^{\beta_1}z_2^{\beta_2}$ is

$$L\left(z_1^{\beta_1}z_2^{\beta_2}\right) = -\frac{\alpha^2}{2}\left[\sigma(\beta_1, \beta_2) + \frac{z_1^2}{z_2^2}(\beta_2 - 1 - 4a)\beta_2 + \frac{z_2^2}{z_1^2}\beta_1(\beta_1 - 1) + \frac{4\beta_1}{z_1}\right]\left(z_1^{\beta_1}z_2^{\beta_2}\right) \tag{50}$$

where constants $\sigma(\beta_1, \beta_2)$ read

$$\sigma(\beta_1, \beta_2) \equiv \beta_1(\beta_1 - 1) + \beta_2(\beta_2 - 1) + 4\beta_1\beta_2 - 4a(2\beta_1 + \beta_2) - 2(2s_0 - 1)(\beta_1 + \beta_2).$$

Using (50) and linear combinations of two terms (50) for different powers (β_1, β_2) and $(\tilde{\beta}_1, \tilde{\beta}_2)$, one can construct only three solutions of (48):

(1) $\Theta^{(1)}(\vec{x}) = z_2^{(4a+1)}$ $\qquad \gamma^{(1)} = \alpha^2(2s_0 - 1)(4a + 1)$ (51)

(2) $\Theta^{(2)}(\vec{x}) = z_2^{(4a+1)}\left(z_1 + \dfrac{2}{4a - 2s_0 + 3}\right)$ $\qquad \gamma^{(2)} = 4\alpha^2(s_0 - 1)(2a + 1)$ (52)

(3) $\Theta^{(3)}(\vec{x}) = z_1 - \dfrac{2}{4a + 2s_0 - 1}$ $\qquad \gamma^{(3)} = \alpha^2(4a + 2s_0 - 1).$ (53)

The eigenvalues γ of (48) can be easily identified, when the corresponding functions $\Theta(\vec{x})$ do not spoil the normalizability of $\Psi_0(\vec{x})$. Within the bounds imposed (39) *only the case (53) is acceptable* provided one also imposes $s_0 > -a + \frac{1}{2}$.

With a similar procedure one can try to construct further eigenfunctions of H based on other $\Psi_k(\vec{x})$; $k > 0$ in turn formed via superpositions (44) of zero modes. Of course, this task will be more difficult than the one we illustrated above, since the coefficient functions of the operator L will contain explicitly the hypergeometric functions from (33). In the particular case $n = 0$, described above, the hypergeometric functions reduce to 1.

5. Shape invariance for two-dimensional systems

In section 4 we studied the two-dimensional spectral problem starting from the linear space spanned by zero modes $\Omega_n(\vec{x})$ of supercharge Q^+ and then constructing an additional eigenstate (53) outside this space by choosing a suitable ansatz. It is obvious that algebraic methods to extend this construction to a larger space of eigenfunctions are highly welcome. In this respect the well-known (in one dimension) method of shape invariance (subsection 5.1) is very suggestive. While in one dimension shape invariance amounts effectively to exact solvability, in two dimensions we will show that one can achieve partial solvability only. The reasons for this will be particularly clear from an analysis of simple two-dimensional systems with separation of variables (subsection 5.2). On one hand, one knows how to solve the problem for this system by standard shape invariance for each degree of freedom. On the other hand, we will also consider this problem directly from a two-dimensional point of view and show that, in general, only partial solvability will be achieved in this last approach. Finally (subsection 5.3), we will extend our method to the two-dimensional systems with SUSY-separation of variables, already described in section 4, where only partial solvability holds.

5.1. One-dimensional shape invariance and solvability

For reader's convenience we write the basic steps of standard shape invariance in the *absence of spontaneous SUSY breaking* [1, 8]. One refers to shape invariance when a one-dimensional superHamiltonian $\hat{H}$ depends on a parameter a and, in addition, its components H and $\tilde{H}$ satisfy:

$$\tilde{H}(a) = H(\bar{a}) + \mathcal{R}(a) \tag{54}$$

where $\bar{a} = \bar{a}(a)$ is some new value of parameter, which depends on a, and $\mathcal{R}(a)$ is a (c-number) function of a. The absence of spontaneous breaking of supersymmetry for all values of a implies that the lowest eigenvalue $E_0(a)$ of $H(a)$ vanishes and the corresponding eigenfunctions $\Psi_0(a)$ are normalizable zero modes of $Q^+(a)$. It is well known [1, 8], that the intertwining relations

$$Q^-\tilde{H}(a) = H(a)Q^-(a) \tag{55}$$

with the standard first-order supercharge (4) allow in this case the entire spectral problem for $H(a)$ to be solved.

The crucial steps are as follows. Starting from

$$H(\bar{a})\Psi_0(\bar{a}) = E_0(\bar{a})\Psi_0(\bar{a}) = 0 \tag{56}$$

consider relation (54) to obtain

$$\tilde{H}(a)\Psi_0(\bar{a}) = \mathcal{R}(a)\Psi_0(\bar{a}). \tag{57}$$

It is important to remark that $\Psi_0(\bar{a}) \equiv \tilde{\Psi}_0(a)$ has no nodes and therefore is the ground state wavefunction of $\tilde{H}(a)$. The combination of (55) and (57) yields

$$H(a)[Q^-(a)\Psi_0(\bar{a})] = \mathcal{R}(a)[Q^-(a)\Psi_0(\bar{a})]. \tag{58}$$

Provided $[Q^-(a)\Psi_0(\bar{a})]$ is normalizable, we have generated an excited state of $H(a)$ and thus $\mathcal{R}(a)$ is naturally positive. It is clear that these steps can be repeated up to the last step, where the resulting wavefunction Ψ will no longer be normalizable. There are notorious cases (oscillator-like potentials) where the spectrum is not bounded from above.

It is also clear that the isospectrality of $H(a)$ and $\tilde{H}(a)$ (up to the only zero mode $\Psi_0(a)$) implies that there is no eigenvalue of $H(a)$ between zero and the ground state energy $\tilde{E}_0(a)$

of $\tilde{H}$. This observation leads to a proof that after suitable iterations one gets the entire spectrum of $H(a)$. This method is referred to as algebraic solvability (or complete solvability) by shape invariance in one-dimensional SUSY QM.

5.2. Two-dimensional shape invariance for systems with separation of variables: solvability or partial solvability

Already the trivial two-dimensional model with separation of variables

$$H(\vec{x}) = H_1(x_1) + H_2(x_2) \qquad \vec{x} = (x_1, x_2)$$

shows that there is a considerable difference with respect to the one-dimensional case. The crucial reason is that the space of zero modes of supercharges now becomes of higher dimensionality including the products of one-dimensional zero modes of the first Hamiltonian and all states of the second Hamiltonian and vice versa.

In order to realize a non-trivial intertwining relation, one can consider factorized supercharges of second order written as products of first-order supercharges[6]:

$$Q^{\pm} = Q_1^{\pm} Q_2^{\pm} \qquad Q_i^{\pm} = (\mp\partial_i + W_i(x_i)). \tag{59}$$

Now suppose that both H_1 and H_2 are shape invariant:

$$\tilde{H}_1(a_1) = H_1(\bar{a}_1) + \mathcal{R}_1(a_1) \qquad \tilde{H}_2(a_2) = H_2(\bar{a}_2) + \mathcal{R}_2(a_2)$$

i.e. H is shape invariant with a vector parameter $\mathbf{a} = (a_1, a_2)$:

$$\tilde{H}(\mathbf{a}) = H(\bar{\mathbf{a}}) + \mathcal{R}(\mathbf{a}).$$

While iterations analogous to subsection 5.1 are obviously possible, it is clear that one cannot argue about the entire solvability of the spectral problem, because in general many zero modes of (59) exist. Their number depends on the confining properties of H_1 and H_2. For example, in the case of oscillator-like potentials this number becomes infinite, and they are distributed over the whole spectrum. In this case only partial solvability of H can be achieved by the choice of (59) and shape invariance. Of course, one can solve such trivial models by separate use of $Q^{\pm} = Q_i^{\pm}$, which allows us to solve the entire spectrum of the two-dimensional model in terms of the one-dimensional ones.

5.3. Shape invariance and partial solvability for two-dimensional systems

Let us suppose to have a two-dimensional system with a Hamiltonian H, which is related to $\tilde{H}$ by (54). For simplicity, we have assumed[7] that shape invariance is realized only with the parameter a. Two-dimensional SUSY QM, realized via (8), (9), does not identify zero modes of $Q^{\pm}$ with the ground state of the Hamiltonian. Thus one has to repeat the steps (54)–(58) by taking into account $E_0(a) \neq 0$. In order to make our discussion more explicit, we will from now on refer explicitly to the model (29)–(32) with the parameter a being bound to (39), as described in section 4.

First of all we observe that this model is indeed shape invariant with

$$\bar{a} = a - \tfrac{1}{2} \qquad \mathcal{R}(a) = \alpha^2(4a - 1). \tag{60}$$

[6] Note that in such models H allows the separation of variables, but $Q^{\pm}$ do not. If one wants $Q^{\pm}$ also to allow the separation of variables, then one would have to consider $Q^{\pm} = Q_i^{\pm}$.

[7] We recall that, in general, there is no connection between the dimensionality of the Schrödinger equation and the dimensionality of the parameter manifold.

We also remark that the infinite domain given by (39) allows the iterations of (54). The starting point is to write (58):

$$H(a)\left[Q^-(a)\Psi_0\left(a-\tfrac{1}{2}\right)\right]=\left(E_0\left(a-\tfrac{1}{2}\right)+\mathcal{R}(a)\right)\left[Q^-(a)\Psi_0\left(a-\tfrac{1}{2}\right)\right] \quad (61)$$

where $E_0(a)$ and $\Psi_0(a)$, not to be identified with the ground state, are given by (44) and (45). Thus we have constructed the new eigenstate and eigenvalue of $H(a)$, provided $Q^-(a)\Psi_0\left(a-\frac{1}{2}\right)$ is normalizable. We note that the eigenvalue $\left(E_0\left(a-\frac{1}{2}\right)+\mathcal{R}(a)\right)$ is larger than $E_0(a)$ with the bounds of (39).

It is interesting to compare this first iteration (61) of shape invariance with the solution (53) obtained in the framework of the ansatz described in subsection 4.6. Their eigenvalues coincide *precisely*:

$$E=E_0(a)+\gamma^{(3)}(a)=E_0\left(a-\tfrac{1}{2}\right)+\mathcal{R}(a)=\alpha^2[4a(1-s_0)+(2s_0-1)]+2\epsilon_0$$

while the eigenfunctions (both vanishing for $x_-=0$) differ by a factor $x_-/|x_-|$ reflecting the opposite values of 'x_--parity'.

The next iteration of shape invariance will give

$$H(a)\left[Q^-(a)Q^-\left(a-\tfrac{1}{2}\right)\Psi_0(a-1)\right]=\left(E_0(a-1)+\mathcal{R}\left(a-\tfrac{1}{2}\right)+\mathcal{R}(a)\right)\times\left[Q^-(a)Q^-\left(a-\tfrac{1}{2}\right)\Psi_0(a-1)\right] \quad (62)$$

and the new eigenfunction $Q^-(a)Q^-\left(a-\frac{1}{2}\right)\Psi_0(a-1)$ can be written explicitly as a function of $\vec{x}$. Provided normalizability is ensured, one can thereby construct a chain by successive iterations of (61) and (62), since $Q^-(a)$ has no normalizable zero modes in (39). The end point of such a chain will be given by non-normalizability of the relevant wavefunction.

6. Discussions and conclusions

We want to point out that SUSY-separation of variables can be implemented completely independently from shape invariance. To this end we present a model where SUSY-separation of variables holds but shape invariance does not apply.

Consider the model (29)–(32) with a choice for parameter *a alternative* to (39):

$$a\in\left(-\tfrac{1}{4},\tfrac{1}{4}\right)\qquad s_0>2(|a|+1). \quad (63)$$

It is obvious that shape invariance does not apply since the domain (63) is too small. One important property of this model is that the reflection symmetry $a\to -a$ is explicitly implemented. This allows the normalizability of both the zero modes[8] $\tilde{\Omega}(a)=\Omega(-a)$ of Q^- and $\Omega(a)$ of Q^+ and also of *all three* wavefunctions $\Phi^{(i)}(\vec{x})$ with $\Theta^{(i)}(\vec{x})$ given in (51)–(53).

Although we have achieved only partial solvability, we now present a short discussion of the quantum integrals of motion (*symmetry operators*), which exist [3, 11, 12] for all HSUSY QM systems. Indeed, the intertwining relations (8) lead to the existence of the symmetry operators $\tilde{R}$, R for the Hamiltonians $\tilde{H}$, H, correspondingly:

$$\tilde{R}=Q^+Q^-\qquad R=Q^-Q^+\qquad [R,H]=0\qquad [\tilde{R},\tilde{H}]=0. \quad (64)$$

In the one-dimensional case [3] these quantum integrals of motion R, $\tilde{R}$ become polynomials of H, $\tilde{H}$ with constant coefficients. The distinguishing peculiarity of the two-dimensional case is given by [11, 12] non-trivial symmetry operators $\tilde{R}$, R which are not reduced to functions of the Hamiltonians H, $\tilde{H}$, i.e. all two-dimensional systems (15) which solve the intertwining relations (8) are *integrable*.

[8] We recall that $\omega_n(\vec{x})$ by construction do not depend on the parameter a (see subsection 4.2).

More specifically, it was shown in [11], in the case of second-order HSUSY, that for the particular case of the unit metrics $g_{ik} = \delta_{ik}$ the fourth-order operators $\tilde{R}$, R can be written as second-order differential operators up to a function of H, $\tilde{H}$ (R-separation of variables [2]). For all other metrics g_{ik} the operators $\tilde{R}$, R are of fourth order in derivatives.

By construction, the quantum integral of motion $R = Q^- Q^+$ gives zero when acting on the eigenfunctions of Hamiltonian Ψ_k from (44). A direct calculation shows that the additional eigenfunctions $\Phi^{(i)}(\vec{x})$ from (51)–(53) are also eigenfunctions of the operator R:

$$R\Phi^{(i)}(\vec{x}) = \alpha^2 r_i \Phi^{(i)}(\vec{x})$$

with corresponding eigenvalues

$$r_1 = -(4a+1)(2s_0-1)(4a-2s_0+1)$$
$$r_2 = -16(2a+1)(s_0-1)(2a-s_0+1)$$
$$r_3 = -(4a-1)(2s_0-1)(4a+2s_0-1).$$

Thus the eigenfunctions (44) and (51)–(53) are part of the system of common eigenfunctions of two Hermitian mutually commuting operators, the Hamiltonian H and the quantum integral of motion R.

In conclusion, we have formulated two new approaches to partial solvability of two-dimensional quantum systems:

(1) The SUSY-separation of variables method (section 3) is to be considered as a particular branch of the method of separation of variables, though not for the Hamiltonian, but for the supercharge. It is in contrast to the more trivial case, where H allows separation, but $Q^{\pm}$ is factorized and does not allow the separation (subsection 5.2).
(2) The method of shape invariance, well known for the one-dimensional SUSY QM, has been reformulated (subsection 5.3) for the two-dimensional systems, which depend parametrically on a, in the same way as for one-dimensional shape invariant systems. This construction is based on knowledge of the eigenfunctions $\Psi_0(a)$ and of the eigenvalues $E_0(a)$ (for a in the domain (39)). We have illustrated this method applying (subsection 5.3) it to a 'singular' two-dimensional Morse system (subsection 4.1) with higher order SUSY QM in the presence of a variety of zero modes of Q^+.

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